Algebraic Structure of Boolean and Cubic Graphs

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Abstract— T he concept of applying Group Theory on Graphs has been developed by many researchers in order to demonstrate very basic and important graph properties and establish the connections between groups and graph theory. This was carried out by many researchers such as Biggs [1], who studied graph theory with Algebraic terms. This gave some principles ideas of applications of Algebra on Graph Theory.

Many researchers, such as [2], [8], studied the automorphisms of graphs and group theory. One of the areas of group theory which has been constantly used in graph theory is the symmetric groups and the action of the group elements on the edges and the vertices of a specific graphs.

Index Terms—Boolean graphs, Cubic graphs, Cyclic groups, Direct product of groups, Isomorphic graphs, Groups of automorphisms, n-dimensional cube, Permutations, Symmetric groups.

INTRODUCTION

THIS section deals with the basic principles of graphs and groups. It introduces the necessary definitions and propertied needed in the second section. These definitions and properties are manly extracted from other references such as [3], [4], and [7]. However, our attention has been restricted to the properties needed in our work in this paper. So we have backed these definitions by various examples to illustrate the main purpose.
The main results are given in Section Two. In this section we have established a connection between the Cubic graph and the symmetric group S_n including the cyclic groups and the direct products of groups.

1 BASIC PROPERTIS

Definition 1.1: A graph consists of two sets V(G) and E(G) called the set of vertices and the set of edges of G, respectively, together with two functions $i: E \rightarrow V$ and $t: E \rightarrow V$.

We say that the edge "e" joins the vertex i(e) to t(e). The vertex i(e) is called the **initial** vertex of "e" and t(e) is called the **terminal** vertex of "e". For each e in E there is an element $\bar{e} \neq e$ in E called the inverse of "e" such that $i(\bar{e}) = t(e), t(\bar{e}) = i(e) and\ \bar{\bar{e}} = e$.

We say that u and v are adjacent to each other if uv is an edge in E(G), moreover, u ~ v if u v is an edge in E (G).
A **walk** is a sequence of vertices and edges such that: $w = v_0e_1v_2e_2 \ldots e_nv_n$, where some edgesand vertices can be visited more than once.

Definition 1.2: Let $v = \{v_0, v_1, \ldots v_k\}$ be a set of pairwise distinct points and let $E = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$. Then $P(V, E)$ is a **path** from $v_0\ to\ v_k$ of length k. So a path of length "k" has k edges and (k+1) vertices.

Definition 1.3: Let $p = p(v_1, v_k)$ be as in the above definition, then $C = C^{k+1} = p + v.v_k$is a cycle of length (k+1). The length of a cycle is the number of vertices or edges on it.

Definition 1.4: A **path** $v_0, e_1, n_2, e_2, \ldots, v_ne_n$ is closed if $v_0 = v_n$. A cycle can be described as a circuit in which no vertex appears more than **once** -except the first which is also the last.

$$v_0e_1v_1e_2v_2e_3v_3e_4v_4e_5v_5e_6v_6$$

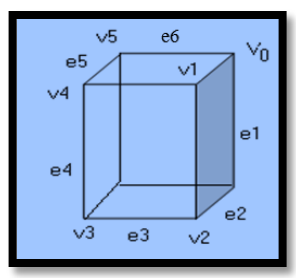

Figure (1)

An **n-cycle** is a cycle with n vertices. A **trail** is a walk with all vertices distinct. A **circuit** is a trail from any vertex back to itself. This means a circuit is a closed walk with all edges distinct.

Definition 1. 5: A graphs $G(V, E)$ is called connected if for any $v_1, v_2 \in V$, there existsa path for $v_1\ to\ v_2$.
We say that u and v are adjacent to each other if uv is an edge in $E(G)$, moreover $u \sim v$ if uv is an edge.

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Definition 1.6: Let $G = (V, E)$ be a graph. If $v \in V$ then $N_G(v) = \{u: vu \in E\}$ is the set of neighbours of v. $d(v) = |N_G(v)|$is the degree of v.

Definition 1.7: If $d(v) = k$, when k is an integer for all $v \in V$ then G is called **regular** of degree k. So the graph is regular or k-regular, if all its vertices have the same degree k.
The following theorem is very well known and given in many other references.

Theorem (Handshake): Let $G = (V, E)$ be a graph. Then $\sum_{v \in V} d(v) = 2E$

Proof Let $X = \{(v, e): v \text{ is an end vertex of } e\}$
Count X in two directions. When we start with e, we will get $|X| = 2|E|$. Starting with v we get $|X| = \sum_{v \in V} d(v)$.

Definition 1.8: Let $G = (V, E)$ and $G' = G(V', E')$ be graphs. Then a bijective map $\emptyset: V \to V'$ is an **isomorphism** provided that $\{u, v\} \in E$ iff $(\emptyset(u), \emptyset(v)) \in E'$. $\emptyset$ is called an **Automorphism** if G= G'.

Definition 1.9: Let $V = \{v_0, v_1, \dots, v_k\}$ be a set of pair wise distinct points, and let $E = \{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\}$.
Then $P = (E, V)$ is a path from v_0 to v_k of length k. So a path of length k has k edges and $(k + 1)$ vertces.

Definition 1.10: Let $P = P(v_0, v_k)$ be as in definition 1.6. Then $C = C^{k+1} = P + v_0v_k$ is a **cycle** of length $(k + 1)$. The length of a cycle is the number of vertices or edges on it.
Definition 1.11: Let $G = (V, E)$ be graph. If v, v' are vertices, then the distance $d_G(v, \acute{v}) = d(v, v')$ from v to v' is the least k so that there is a path $P(v,\ v') = P^k \subseteq G$.

The **diameter** diam (G) of G is the largest k such that $d(v, v') = k$ for some pair of vertices. The **girth** $g(G)$ of G is the least k such that $C^k \subseteq G$ for some cycle C^k.

2. SYMMETRIC GROUPS

Definition 2.1 A permutation of a set S is a one-to-one, onto mapping from S to itself.

Let S $= \{1,2,3\}$, the permutations of S are: e = (1)(2)(3), a = (1)(23), b = (2)(13), c = (3)(12), d = (123), f = (132)

The set $\{e, a, b, c, d, f\}$ forms a group under composition called the symmetric group S_3 . S_n contains$n!$ elements.
The elements of S_3 can be expressed by :

e = (1)(2)(3), a = (1)(23), b = (2)(13), c = (12)(3), d = (123), f = (132), where $a^2 = b^2 = c^2 = 1$ and $f^{-1} = (2\ 3\ 1) = d$.

A permutation $\pi \in S_n$ which interchanges two elements and fixed all the others is called a **transposition.**

Definition 2.2 A Permutation π in S_n is called cycle if it has at most one orbit containing more than one element. The **length of a cycle** is determined by the length of its largest orbit.

Eg: $\pi = (1)(2,3)(4)$ is a cycle of length 2.

Let π = $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{bmatrix}$

Then π is split into the following orbits:
{ 1, 2, 4 }, { 3, 5 }, { 6 }

A permutation of a finite set is even or odd according to whether it can be expressed as a product of an even number of transpositions or the product of an odd number of transpositions.

Example
$(1, 4, 5, 6)(2, 1, 5) = (1, 6)(1, 5)(1, 4)(2, 5)(2, 1)$
So this is an odd permutation.
Definition 2.3 The direct product of the groups $G_1, \dots, G_n$ when n is finite is defined by:

$G_1 * \dots * G_n$consisting of elements of the form $(g_1, \dots, g_n)$ where $g_i \in G_i, 1 \le i \le n$.

Definition 2.4 A homomorphism from a group G to a group $\grave{G}$ is a mapping $f: G \to \grave{G}$where $f(g_1g_2) = f(g_1)f(g_2)$, where $g_1,\ g_2 \in G$

Definition 2.5 An isomorphism is a one-to-one onto homomorphism.

Definition 2.6 An automorphism of a group G is an isomorphism of a group G onto itself.
The set of all automorphisms of a group G is denoted by $\boldsymbol{Aut(G)}$.

Theorem 2.1 Let G be a group and $Aut(G)$ be the set of all automorphisms of G.Then $Aut(G)$ forms a group under the compositions of functions.

Proof See [5]

Definition 2.7 Let $G = (V, E)$ be a finite graph. An automorphism of G is a permutation of the vertex set that satisfies the condition $\{u_i, u_j\} \in E(G)$ if and only if $\{\emptyset(u_i), \emptyset(u_j)\} \in E(G)$.

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The Automorphism Group of G is the set of permutations of the vertex set that preserve adjacency. See [3].

$$Aut(G) = \{\pi \in Sym(v): \pi(E) = E\}.$$

Theorem 2.2 The set $Aut(G)$ of all group automorphisms of a group G forms a group under compositions of functions.

Proof See [3].

3. Applications

Definition 3.1 An edge automorphism on a graph $G = (V, E)$ is a permutation π on the set of edges of $G(E)$ satisfying e_i, e_j are adjacent if and only if $\emptyset(e_i), \emptyset(e_j)$ are also adjacent.

Theorem 3.1 $Aut(K_n) \simeq S_n$ (isomorphic).
Proof K_n contains n vertices which are all connected to each other. Each vertex is connected to $n-1$edges . Each vertex from K_n is mapping to another vertex. The other $n-1$ vertices are connected to $n-2$ vertices and so on .

Therefore, $Aut(K_n)$ contains $n(n-1)(n-2)\dots 2\times 1$ elements and this given by $n!$. Since S_n contains $n!$ Elements, so each element in $Aut(K_n)$ is mapped to an element in S_n.
Therefore, $Aut(K_n) \simeq S_n$
The following is covered in [6]

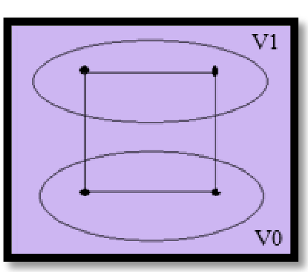
Cube Graph: Let V be the set of $v = (x_1, x_2, \dots, x_n)$ with $x_i \in \{0,1\}$ and $n > 0$. $v_1 v' \in V$, are adjacent if and only if they differ by only one coordinate when expressed as vectors . ie $(x_1, \dots, x_n)$ is adjacent to $(x'_1, \dots, x'_n)$ if and only if they differ in precisely one position ,
These will form the edges .
The Graph $G = C_u^n,\ n > 0$. (n-dimensional graph).

$$V_0 = \{(0, x_2, \dots, x_n) \quad x_i \in \{0,1\}$$

$$V_1 = \{(1, x_2, \dots, x_n) \ x_j \in \{0,1\}$$

$$V = V_0 \cup V_1$$

$$V = V_0 \cup V_1 and\ V = V_0 \cap V_1 = \emptyset$$



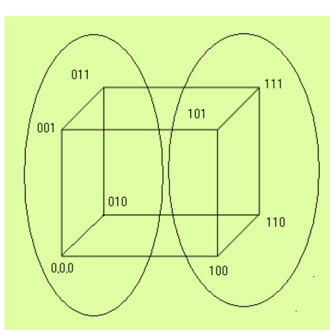


Figure (2)

Theorem 3.2 Every cube graph C_u^n has a Hamiltonian cycle if ≥ 2 .
Proof: See [6].

Let v be the set of vertices of a graph. A permutation of V is an automorphism of a graph G if the following conditions hold: $uv \in E$ if and only if $(u)g(v) \in E$.
Now let $C_2^n = c_2 \times c_2 \times \dots \times c_2$ be the direct product of n copies of cyclic groups of order 2.

Theorem 3.3: C_2^n acts on C_u^n as a group of outomorphisms .
Proof: See [6].

The Group of Automorphisms of C_u^n

C_2 can be regarded as a group of integers Z_2
Therefore $C_2^n = c_2 \times c_2 \times \dots \times c_2$ is the set of all sequences $g = (g_1, g_2, \dots, g_n)$ with group operation $g + g' = (g_1, g_1' , g_2, g_2' , \dots, g_n, g_n')$. Identify the vertices of C_u^n with the elements of C_2^n .
If $g \in C_2^n$ where $x = (x_1, \dots, x_n)$ is a vertex of C_2^n, then let $g = (g_1 + x_1, \dots, g_n + x_n)$. This is a one-to-one automorphism map of the vertex set where x and x' differ in exactly one position if g(x) and g(x') differ in one position .

Boolean Graphs : Let X={1,2, … ,n} . Take V as the collection of subsets of X. $v_1 \neq v_2 \in V$ are adjacent to each other if $(u \backslash v) \cup (v/u)$ has only one element. This graph is called the Boolean graph B^n.

Now let G = (V, E) = B^n be a Boolean graph. Let V_o be the collection of all subsets of { 1, 2, 3,…, n-1 } and let V_1 be the collection of all subsets of { 1, 2, 3, …,n }.
Therefore $V = V_o \cup V_1$ is the set of vertices of B^n. If $u, v \in V_i$ we define $u \sim v$, if and only if the symmetric difference between $u\ and\ v$ has size one. Then we get the graph $G_i = (V_i, E_i)$ will be isomorphic to B^{n-1}

There are edges between $x \in V_o\ and\ y \in V_1$ if and only if $x = y \cup \{n\}$. So G = B^n and G[V_i] is isomorphic to B^{n-1}.

Now look at the number of vertices and edges in B^n.
From above we get that |V| = |V_o| + |V_1| where |V_o| = |V_1|
By induction we get |V| = 2^n . So there are 2^n subsets of { 1, …,n }
So for any set $u \in V$, there will be n sets which have symmetric difference with u of size 1.
Removing i from u if it is contained in u gives an adjacent or adding i to u if it is not in u gives a neighbor. Therefore the degree of each vertex in G = B^n is n.

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Using Handshake lemma we get n|V| = 2|E| aC_2^nnd so |E| = n 2^{n-1}.

Theorem 3.4 B^n is isomorphic to C_u^n

Proof Let $\emptyset : V(C_u^n) \to V(B^n)$ be a map defined by: $x \sim y \ in \ C_u^n$ if and only if $\emptyset(x) \sim \emptyset(y)$.
So let $x = (x_1, \ldots, x_n) \ with \ x_i \in \{0, 1\}$ be a vertex of C_u^n.

Define $\emptyset(x)$ to be the set $\{i : x_i = 1\}$. So $\emptyset$ is onto.
If $\emptyset(x) \sim \emptyset(y)$ then $x = y$. Therefore $\emptyset$ is a bijection between $V(C_u^n) \ and \ V(B^n)$.

Now let $x, y \in V(C_u^n)$, then $x \sim y$ if and only if $1 = |\{i : x_i \neq y_i\}|$ if and only if the symmetric difference of $\emptyset(x) \ and \ \emptyset(y)$ has size one if and only if $\emptyset(x) \sim \emptyset(y) \in V(B^n)$.
Therefore $\emptyset : V(C_u^n) \to V(B^n)$ is an isomorphism.

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